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PARTICULAR SOLUTIONS FOR FLOWS AT MACH NUMBER 1

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SUMMARY

The small-disturbance equation for flow at Mach number 1 is studied with the objective of determining closed analytic solutions representing possible fluid motions. Two families of exact solutions appear. The first, which represents purely supersonic flow, is, in the two-dimensional case, the transonic equivalent of the Prandtl-Meyer expansion and leads naturally to simple wave systems over two-dimensional surfaces. The companion solution, in the case of rotational symmetry, yields a conical field which coalesces onto the axis. A second family of subsonic solutions is also calculated. In the case of rotational symmetry, a source-like flow with constant mass flux in the downstream half-plane results.

It is possible to patch these solutions together in such a way as to simulate the flow over the rear half of an infinite body. When initial conditions on the body dictate sonic speed, expansion waves first appear and accelerate the motion. An abrupt change to the subsonic regime is then produced by a single shock of parabolic shape.

INTRODUCTION

This paper presents a few exact solutions of the transonic differential equation for the perturbation potential when the free-stream Mach number is 1. Such examples, as in the case of Ringleb's solution (ref. 1) of the more general potential equation, are of some interest if only by virtue of their rarity, and one can hope that they may provide some insight into the possibility of introducing further simplification in the methods used to solve the basic nonlinear equation. The approach used here is effectively a separation of variables technique with the additional assumption that the perturbation potential times some power of lateral distance is a function of a single variable that is related simply to lateral and longitudinal distance. Since no boundary conditions are imposed, it becomes relatively easy to generate solutions and one finds that both two-dimensional and axially symmetric flow fields can be treated in the same manner.

The solutions yield, in all cases, either purely supersonic or purely subsonic flow. The method of analysis, in fact, thwarts all attempts to attain a mixed type of solution except for the case in which initially supersonic flow is joined by means of a shock wave onto the downstream portion of the subsonic solution. The details of the joining are given in the final section on applications.

IMPORTANT SYMBOLS

a speed of sound

C_p pressure coefficient, $\frac{p - p_o}{\frac{1}{2} \rho_o U_o^2}$

k coefficient in transonic potential equation (see eq. (10))

M Mach number, $\frac{V}{a}$

p pressure

u,v,w perturbation velocity components in x,y,z directions

U total velocity component in free-stream direction

V total velocity

x,y,z Cartesian coordinates (if axial symmetry is present, y is radial coordinate, see eq. (13))

γ ratio of specific heats, for air $\gamma = 1.4$

δ_{max} maximum angle of flow deflection for attached shock wave

ρ density

τ composite variable (see eq. (15))

Φ velocity potential function

φ $U_o^{-1}\Phi$

Subscripts

a evaluated just ahead of a shock wave

b evaluated just behind a shock wave

o evaluated in the free stream

* evaluated at critical speed ($M = 1$)

ANALYSIS

In this section, analysis basic to the determination of the exact solutions will be presented for two-dimensional and axially symmetric flow fields when the free-stream Mach number is 1. The derivation of the transonic partial differential equation is first reviewed briefly. Particular restrictions on the structure of the fields and the free-stream Mach number then lead to consideration of an ordinary differential equation that is solvable analytically. The section concludes with a discussion of the properties of the derived solutions.

Basic Equations

The flow to be studied is, by assumption, produced by slight deviations in the uniform flow field created by a free stream of velocity U_o. The uniform flow will be directed along the positive x axis of a Cartesian coordinate system x,y,z and the perturbation velocity components u,v,w, parallel respectively to the three coordinate axes, are assumed small relative to U_o and the speed of sound a_o in the stream. For transonic flow, the free-stream Mach number ($M_o = U_o/a_o$) is near 1 and, in the derivation of the governing potential equation, the nonlinearity is retained only insofar as the streamwise component of perturbation velocity affects the result. The continuity equation can thus be approximated in the form

$$\frac{\partial}{\partial x}\left[(U_o+u)\rho(U_o+u,0,0)\right]+\frac{\partial}{\partial y}(v\rho_o)+\frac{\partial}{\partial z}(w\rho_o) = 0 \tag{1}$$

where $\rho(U_o+u,v,w)$ is density expressed as a function of local velocity and $\rho_o = \rho(U_o,0,0)$ denotes free-stream conditions. The first term of equation (1) represents the gradient of mass flow and because of the unidimensionality is easily evaluated by means of the corresponding momentum relation

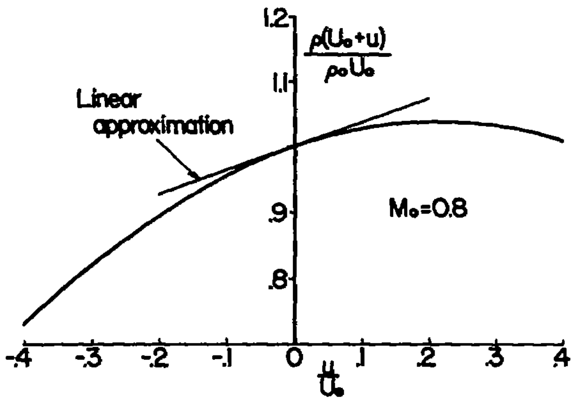
$$U_o+u = -\frac{1}{\rho}\frac{dp}{du} \tag{2}$$

where ρ is $\rho(U_o+u,0,0)$ and p is local pressure. Equation (2), together with the isentropic pressure-density relation $p/p_o = (\rho/\rho_o)^\gamma$ yields

$$\frac{p}{p_o} = \left\{1 + \frac{\gamma - 1}{2} M_o^2\left[1 - \left(1 + \frac{u}{U_o}\right)^2\right]\right\}^{\gamma/(\gamma-1)} \tag{3}$$

where γ is the ratio of specific heats. The mass flow is, thus, from equation (2), determined by

$$\frac{(U_o + u)\rho}{U_o\rho_o} = \left(1 + \frac{u}{U_o}\right)\left\{1 + \frac{\gamma - 1}{2} M_o^2\left[1 - \left(1 + \frac{u}{U_o}\right)^2\right]\right\}^{1/(\gamma-1)} \tag{4}$$



Sketch (a)

and it remains to evaluate its streamwise gradient.

Sketch (a) shows a plot of equation (4) when $M_o = 0.8$. As is to be expected, the maximum mass flow occurs when the local Mach number is 1. The portion of the curve in the neighborhood of this maximum is the part presently of interest. If equation (4) is approximated by an expansion including only terms of first order in u/U_o, the mass flow is represented by a straight line tangent to the curve at the point $u = 0$, as shown in the sketch. Equation (1) then assumes the form associated with linearized compressible-flow theory. A more exact approximation is sought here, and to this end the quadratic dependence on u/U_o is retained. The mass-flow curve is thus to be represented by a parabola of the form

$$\frac{(U_o + u)\rho}{U_o\rho_o} = 1 + (1 - M_o^2) \frac{u}{U_o} - \frac{U_o k}{2}\left(\frac{u}{U_o}\right)^2 \tag{5}$$

where k is a constant yet to be determined.

In equation (5) the ordinate and slope of the parabolic approximation correspond to free-stream conditions but some arbitrariness remains in the determination of k. The most obvious choice would appear to dictate higher-order contact between the two curves, and this leads to the value

$$U_o k = M_o^2[3 - (2 - \gamma)M_o^2] \tag{6}$$

A more interesting possibility follows if the vertex of the approximating parabola is fixed at $M = 1$, although the value of the mass flow itself may not be exact there. This is the approach used by Oswatitsch (ref. 2) and leads to the result

$$U_0 k = U_0 \frac{(1 - M_0{}^2)}{a_* - U_0} \tag{7}$$

where a_* is the critical speed of sound determined by $M = (U_0 + u_*)/a_* = 1$.

A third possibility for the determination of k follows from the known relation

$$\frac{a_*}{U_0} = \left[1 + 2\,\frac{(1 - M_0{}^2)}{(\gamma + 1)M_0{}^2}\right]^{1/2} \tag{8}$$

If equation (7) is combined with the first-order expansion of equation (8) near $M_0 = 1$, one gets

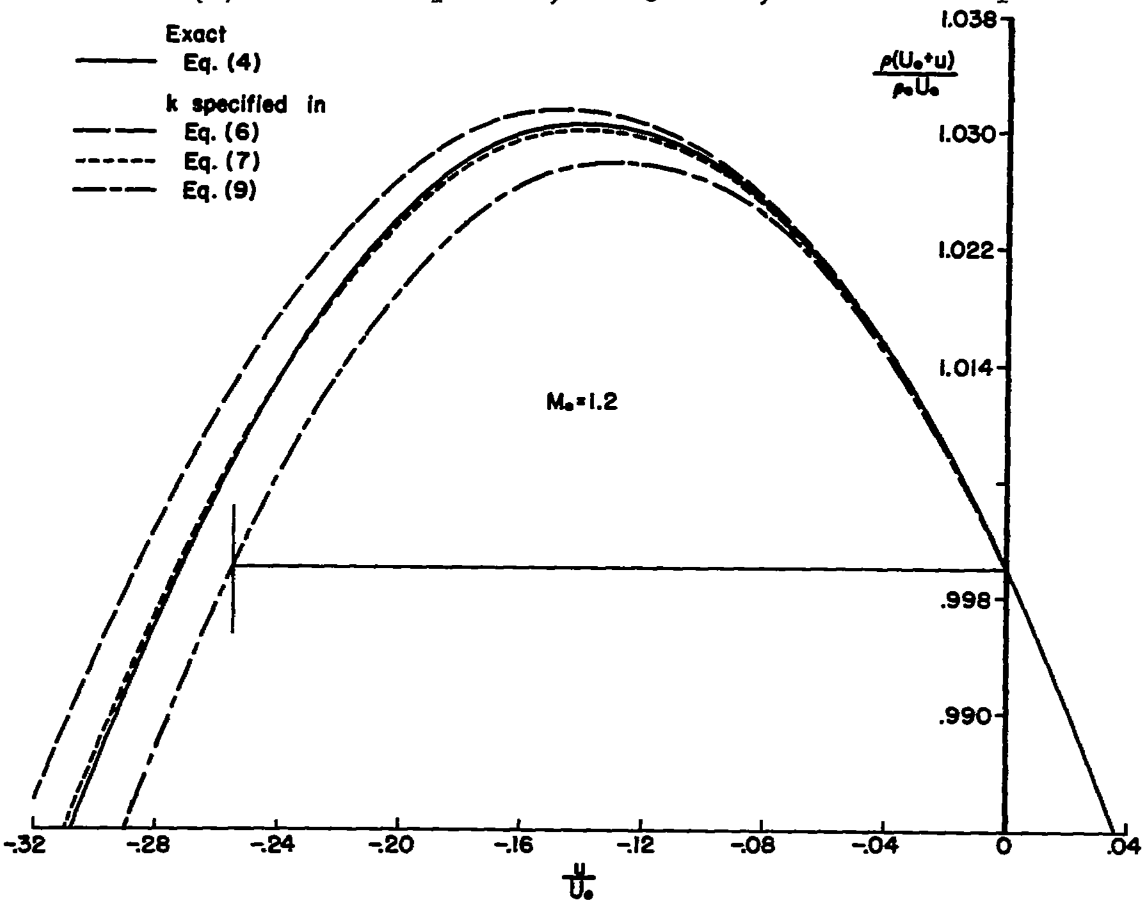
$$U_0 k = (\gamma + 1)M_0{}^2 \tag{9}$$

When equation (5) is used, together with equation (1), and a perturbation velocity potential $\Phi(x,y,z)$ is introduced, the transonic differential equation takes the form

$$\Phi_{xx}(1 - M_0{}^2 - k\Phi_x) + \Phi_{yy} + \Phi_{zz} = 0 \tag{10}$$

where k is given by equation (6), (7), or (9).

Sketch (b) shows a comparison, at $M_0 = 1.2$, of the three parabolas



Sketch (b)

described above, as well as the more exact curve for mass flow given by the isentropic relation of equation (4). The differences between the curves are of the same order of magnitude as those of terms neglected in transonic theory and the final choice must be based on additional considerations. The form of k in equation (9) was proposed originally by Spreiter in reference 3, and its superiority in correlating experimental data by means of the transonic similarity rules was clearly established at that time. Its use will be adopted here for this reason and also because, as shown in the following paragraph, it appears as the most natural form to relate the transonic shock-polar and mass-flow relations.

Equation (10) applies to transonic flow in regions for which the flow experiences no discontinuities and the gradients of velocity in the equation are finite. Transition through a shock wave is governed by a finite-difference relation between the components of perturbation velocity (u_a,v_a,w_a) ahead of the shock wave and the components (u_b,v_b,w_b) behind the shock wave. As given in reference 3 this equation is the transonic shock polar

$$(1-M_o^2)(u_a-u_b)^2+(v_a-v_b)^2+(w_a-w_b)^2 = M_o^2\,\frac{(\gamma+1)}{U_o}\left(\frac{u_a+u_b}{2}\right)(u_a-u_b)^2$$

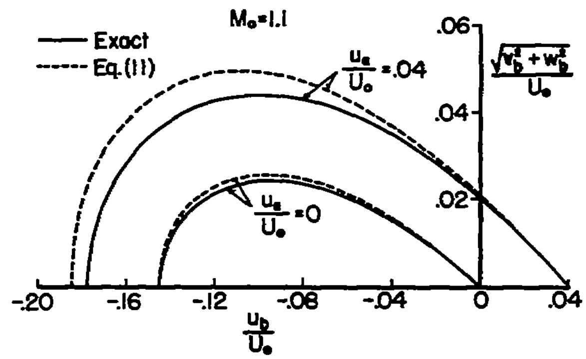
$$= k\left(\frac{u_a+u_b}{2}\right)(u_a-u_b)^2 \qquad (11)$$

where k in the final term is as defined in equation (9). Consider a normal shock wave in the flow with velocity U_o ahead and U_o+u_b behind. Since the mass flow is continuous through the shock wave and since in transonic theory the continuous portion of the flow is represented always as a potential field with uniform stream conditions, it follows that when u_a,v_a,v_b vanish, the value of u_b downstream of the shock must correspond to unity on the mass-flow parabola. In sketch (b), unit mass flow is represented by a horizontal line through $u = 0$. The intersection of the line and the parabola corresponding to the k of equation (9) is at

$$\frac{u_b}{U_o} = 2\,\frac{(1-M_o^2)}{(\gamma+1)M_o^2}$$

and this agrees with the value of u_b given by equation (11). This result can, in fact, be generalized easily to show that for any normal shock the discontinuity predicted by the shock polar agrees with the discontinuity given by equation (5) when the mass flow is held fixed.

A comparison between the transonic shock polar of equation (11) and the exact polar is shown in sketch (c) for the case $M_o = 1.1$. The inner curves apply to shock discontinuities for which free-stream conditions hold on the upstream face; the two polars are thus applicable, in particular, to bow waves in a supersonic free stream. It is theoretically possible to treat shocks at positions for which the supersonic free stream has been accelerated to a higher Mach number associated with the longitudinal perturbation velocity u_a. The outer curves of the sketch are drawn for the case $M_o = 1.1$ and $u_a/U_o = 0.04$. It is obvious that u_a/U_o must, in general, be kept very small if the shock analysis is to be based on free-stream conditions.



Sketch (c)

From equation (11) the maximum deflection angle accommodated by the shock wave can be shown to be

$$\delta_{max} = \frac{4}{3\sqrt{3}} \frac{(M_a^2 - 1)^{3/2}}{M_o^2(\gamma + 1)} \tag{12}$$

where M_a is the Mach number on the upstream face of the shock. It is clear from an inspection of the curves in sketch (c) that this result provides excellent agreement with exact theory when $M_o = M_a$ but becomes less accurate as the difference between the Mach numbers increases.

Particular Solutions When $M_o = 1$

Subsequent application will be limited to cases where the flow field is either two-dimensional or has axial symmetry. The differential relation for the perturbation potential is, from equation (10),

$$(1 - M_o^2 - k\Phi_x)\Phi_{xx} + \Phi_{yy} + \frac{\sigma}{y}\Phi_y = 0 \tag{13}$$

where $\sigma = 0$ for two-dimensional flow and $\sigma = 1$ for axially symmetric flow and y is a Cartesian coordinate in the two-dimensional case and the radial coordinate in the case of axial symmetry. When the free-stream Mach number is 1, all values of k reduce to

$$k_o = (\gamma + 1)/U_o$$

and by means of the transformation

$$\Phi = U_o\varphi$$

the differential equation becomes

$$-(\gamma+1)\varphi_x\varphi_{xx} + \varphi_{yy} + \frac{\sigma}{y}\varphi_y = 0 \tag{14}$$

Following Guderley and Yoshihara (ref. 4) a solution is sought of the form

$$\varphi = \frac{1}{y^m} f(\tau) \ , \qquad (k_oU_o)^{1/3}\tau = \frac{x}{y^n} \tag{15}$$

If the resulting ordinary differential equation is to involve only τ and $f(\tau)$, the condition $m + 3n = 2$ must apply. The perturbation velocity components then become

$$\varphi_x = \frac{\tau}{xy^{2-3n}} f'(\tau) \tag{16a}$$

$$\varphi_y = -\frac{(2-3n)}{y^{3(1-n)}}\left[f(\tau) + \frac{n}{(2-3n)}\tau f'(\tau)\right] \tag{16b}$$

The ordinary differential equation for $f(\tau)$ can now be written as

$$f'f'' = [3(1-n)-\sigma](2-3n)f + [5n(1-n)-n\sigma]\tau f' + n^2\tau^2 f'' \tag{17a}$$

This is a homogeneous differential equation, of degree of homogeneity 3, so its order may be reduced by unity (see ref. 5, par. 55). An alternate approach is possible, however, since the parameter n is yet to be fixed. Rewrite equation (17a) as

$$\frac{1}{2}\frac{d}{d\tau}[f'(\tau)]^2 = n^2\frac{d^2}{d\tau^2}(\tau^2 f) + [5n(1-n) - 4n^2 - \sigma n]\frac{d}{d\tau}(\tau f) +$$

$$[2(1-n)(3-7n) + 2n^2 + 2\sigma(2n-1)]f \tag{17b}$$

A first integral of this equation can be written immediately if n is chosen so as to make the coefficient of f vanish. Solution of the resulting quadratic equation in n yields the roots

$$n = \frac{1}{2}, \quad \frac{3-\sigma}{4}$$

For the two-dimensional case, this results in two possible values for n, namely 1/2 and 3/4; but for the axisymmetric case, where $\sigma = 1$, the roots coalesce at $n = 1/2$. The first integral of equation (17b) is

$$f'^2 = 2n^2 \frac{d}{d\tau}(\tau^2 f) + 2n(5 - 9n - \sigma)\tau f + \text{const}$$

In order to deal with situations in which the flow is undisturbed for $x < 0$, the condition that the streamwise perturbation velocity component φ_x vanish at $x = 0$ will be applied. From equation (16a), this implies that $f'(0) = 0$ also and the value of the constant in the last equation becomes zero. The resulting form for the first integral of equation (17a) is therefore

$$f'^2 - 2n^2\tau^2 f' - 2n(5 - 7n - \sigma)\tau f = 0 \tag{18}$$

It will be noted that equation (18) is again homogeneous, and thus can be integrated. It remains then to consider separately the three possible cases ($\sigma = 0$; $n = 1/2$, $3/4$ and $\sigma = 1$; $n = 1/2$) and to examine the form and properties of the relevant solutions of equation (18).

Properties of the Particular Solutions

Two-dimensional case, $n = 1/2$.- Setting $\sigma = 0$, $n = 1/2$ in equation (18), one needs to solve the equation

$$2f'^2 - \tau^2 f' - 3\tau f = 0 \tag{19}$$

and this is readily carried out by means of the substitution given in reference 5. First, however, consider the equation written in the form

$$f' = \frac{1}{4}\left(\tau^2 \pm \sqrt{\tau^4 + 24\tau f}\right) \tag{20}$$

It can be seen that $f(\tau)$ must be an odd function of τ, positive when τ is positive. The ambiguity of sign then leads to two possibilities: if the upper sign is chosen, $f' > 0$, and the choice of the lower sign leads to $f' < 0$. By reference to equation (16a), it follows that these possibilities lead respectively to the conclusion that $\varphi_x > 0$ or $\varphi_x < 0$, hence to purely supersonic or purely subsonic perturbation fields, since the stream velocity is sonic. These possibilities will be distinguished with respect to f by designating them respectively as f_1 and f_2.

The solutions of equation (19) are expressible in the forms

$$f_1(\tau) = \frac{1}{3}\tau^3 + \frac{f_{10}^{2/3}}{2(3)^{1/3}} \frac{(\tau^3 + 24f_1)^{1/2} + 3\tau^{3/2}}{\left[(\tau^3 + 24f_1)^{1/2} + \tau^{3/2}\right]^{1/3}} \tag{21a}$$

$$f_2(\tau) = 24f_{20}^2 \frac{(\tau^3 + 24f_2)^{1/2} + \tau^{3/2}}{\left[(\tau^3 + 24f_2)^{1/2} + 3\tau^{3/2}\right]^3} \tag{21b}$$

where the additional subscript o means that the function in question is evaluated at $\tau = 0$. A particularly interesting solution of supersonic type results when $f_{10} = 0$, namely

$$f_1(\tau) = \frac{1}{3}\tau^3 \tag{22}$$

Physical aspects of the flow field determined by this solution will be considered later. Supersonic-type solutions given by equation (21a) that do not have the property $f_{10} = 0$ will not be considered further here. It is apparent, however, that they behave asymptotically like $(1/3)\tau^3$, the solution to be used.

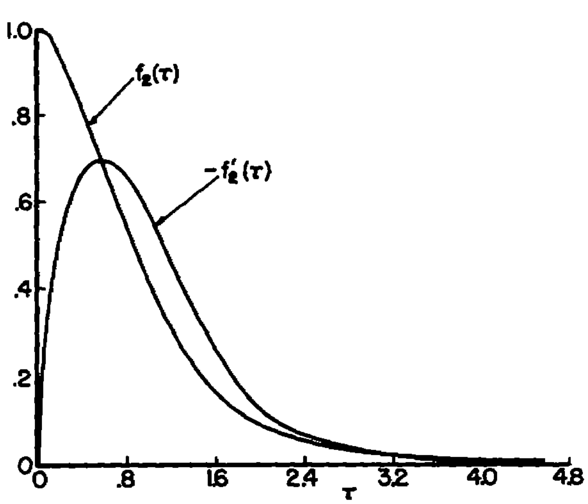
Consider, next, the solution (21b) which is of subsonic type, $f_2' < 0$. The formulation, as given, is implicit but it is not difficult to compute the course of $f_2(\tau)$ because of the following relation, a consequence of the homogeneity. Let $\theta = 24f_2(\tau)/\tau^3$; then equation (21b) can be written

$$\tau^6 = f_{20}^2 \frac{576}{\theta} \frac{(1+\theta)^{1/2} + 1}{\left[(1+\theta)^{1/2} + 3\right]^3} \tag{23}$$

so that if values of θ are assigned, τ may be computed and hence f_2, since

$$f_2(\tau) = \frac{\tau^3\theta}{24}$$

Sketch (d) shows the behavior of $f_2(\tau)$ for the initial value $f_{20} = 1$. Because of the homogeneity of $f(\tau)$



Sketch (d)

in τ, it is only necessary to compute $f_2(\tau)$ for one initial value f_{20}; other cases then follow from the transformations $\tau = c\tau^*$, $f_2(\tau) = c^3 f_2^*(\tau^*)$.

For later analysis, it is convenient to have simple expressions for the behavior of $f_2(\tau)$ for small and for large values of the argument τ. Such expressions are readily calculated by use of the defining equation (19). The required formula for small and large τ are, respectively,

$$f_2(\tau) \approx f_{20} - \left(\frac{2}{3} f_{20}\right)^{1/2} \tau^{3/2} + \frac{1}{4} \tau^3 - \frac{1}{72\left(\frac{2}{3} f_{20}\right)^{1/2}} \tau^{9/2} + \ldots \tag{24a}$$

$$f_2(\tau) \sim \frac{3}{4} f_{20}^2 \frac{1}{\tau^3} - \frac{27}{16} f_{20}^4 \frac{1}{\tau^9} + \ldots \tag{24b}$$

Two-dimensional case, $n = 3/4$.- The function $f(\tau)$ is now replaced by $g(\tau)$ in order to distinguish the present functions from those for $n = 1/2$. The defining equation is

$$8g'^2 - 9\tau^2 g' + 3\tau g = 0 \tag{25a}$$

Solving for g', it follows that

$$g' = \frac{1}{16}\left[9\tau^2 \pm (81\tau^4 - 96\tau g)^{1/2}\right] \tag{25b}$$

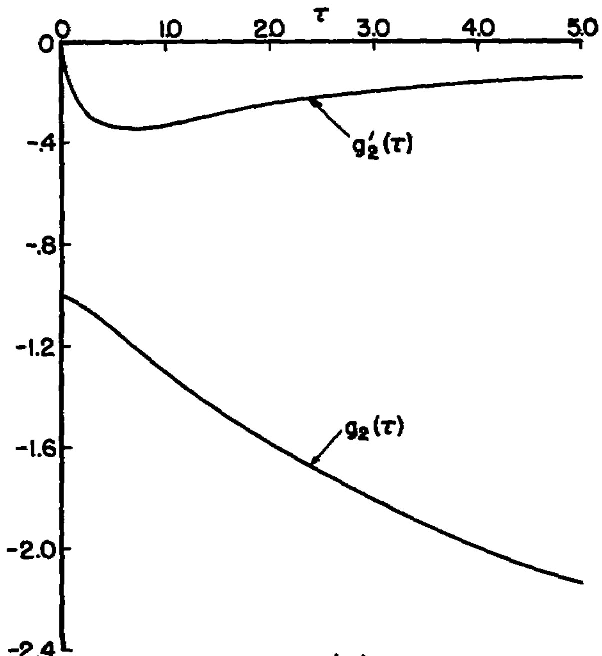
Thus $g(\tau)$ must be negative, or such that

$$g < \frac{81}{96} \tau^3$$

The solutions associated with the two signs in equation (25b) will again be distinguished by the subscripts 1 and 2, corresponding to the positive and negative signs, respectively. These solutions are

$$g_1(\tau) = \frac{1}{3} \tau^3 - (96 g_{10}^8)^{1/7} \frac{(81\tau^3 - 96 g_1 \tau)^{1/2} + 7\tau^{3/2}}{\left[(81\tau^3 - 96\tau g_1)^{1/2} + 9\tau^{3/2}\right]^{9/7}} \tag{26a}$$

$$g_2(\tau) = -\left(\frac{g_{20}^8}{96}\right)^{1/9} \frac{(81\tau^3 - 96\tau g_2)^{1/2} + 9\tau^{3/2}}{\left[(81\tau^3 - 96\tau g_2)^{1/2} + 7\tau^{3/2}\right]^{7/9}} \tag{26b}$$



Sketch (e)

It is of interest to remark that the supersonic-type solution for which $g_{10} = 0$ coincides with that found in the previous case. The solution of subsonic type is shown in sketch (e); it does not vanish for large τ as did $f_2(\tau)$ but, in fact, grows as $\tau^{1/3}$. Approximate formulas for small and large τ are again easy to find, being

$$g_2(\tau) \approx g_{20} - \left(\frac{-g_{20}}{6}\right)^{1/2} \tau^{3/2} + \frac{7}{48}\tau^3 + \ldots \tag{27a}$$

$$g_2(\tau) \sim -\frac{9}{4}\left(\frac{g_{20}{}^8}{192}\right)^{1/9} \tau^{1/3} - \frac{3}{16}\left(\frac{g_{20}{}^8}{192}\right)^{2/9} \frac{1}{\tau^{7/3}} + \ldots \tag{27b}$$

Axisymmetric case.- For this case the functional symbol $F(\tau)$ is used to replace $f(\tau)$. The defining differential equation is

$$2F'^2 - \tau^2 F' - \tau F = 0 \tag{28}$$

which can be rewritten as

$$F' = \frac{1}{4}\left[\tau^2 \pm \left(\tau^4 + 8\tau F\right)^{1/2}\right] \tag{29}$$

Solutions of equation (29) are

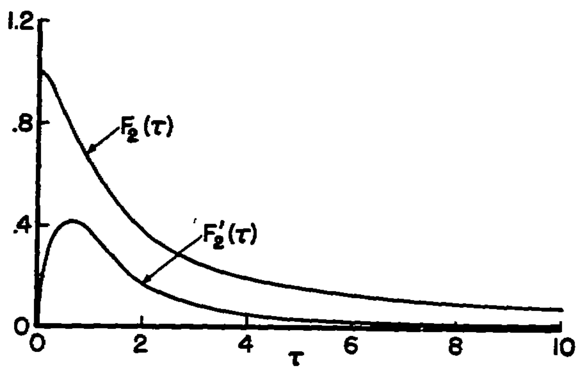
$$F_1(\tau) = \frac{2}{9}\tau^3 + \frac{1}{3}\left(\frac{F_{10}{}^4}{8}\right)^{1/5} \frac{3\left(\tau^3 + 8F_1\right)^{1/2} + 5\tau^{3/2}}{\left[\left(\tau^3 + 8F_1\right)^{1/2} + \tau^{3/2}\right]^{3/5}} \tag{30a}$$

$$F_2(\tau) = 6(9F_{20}{}^4)^{1/3} \frac{(\tau^3 + 8F_2)^{1/2} + \tau^{3/2}}{\left[3(\tau^3 + 8F_2)^{1/2} + 5\tau^{3/2}\right]^{5/3}} \qquad (30b)$$

Once again a simple, supersonic-type solution results when $F_{10} = 0$,

$$F_1 = \frac{2}{9}\tau^3 \qquad (31)$$

The subsonic solution can be calculated by the same device used previously and has the form shown in sketch (f). Expansions for small as well as large values of τ are



Sketch (f)

$$F_2(\tau) \approx F_{20} - \frac{2}{3}\left(\frac{F_{20}}{2}\right)^{1/2}\tau^{3/2} + \frac{5}{36}\tau^3 - \frac{5}{432}\left(\frac{2}{F_{20}}\right)^{1/2}\tau^{9/2} + \ldots \qquad (32a)$$

$$F_2(\tau) \sim \frac{3}{8}(9F_{20}{}^4)^{1/3}\frac{1}{\tau} - \frac{27}{128}(3F_{20}{}^8)^{1/3}\frac{1}{\tau^5} + \ldots \qquad (32b)$$

The formal similarity between the two- and three-dimensional cases is particularly striking for the simple, supersonic-type solutions since they differ only in the magnitude of the coefficient of τ^3. It will be seen in the next section, however, that the two-dimensional solution is of more general interest since it is directly related to the study of expansion waves in transonic flow.

SPECIAL APPLICATIONS

In this section streamlines and pressures associated with the derived solutions will be studied in some detail. In all cases the flow in the negative half-plane $(x < 0)$ will be assumed uniform with parallel streamlines. The disturbance fields then produce deviations in slope of the streamlines in the positive half-plane $(x \geq 0)$ and the flow can thus be identified with modifications in shape on the rear of a semi-infinite cylinder or airfoil.

Two-Dimensional Flow

Supersonic-type solution.- Consider first the supersonic solution determined by the function $f_1(\tau)$ given in equation (22): $f_1(\tau) = \tau^3/3$. As mentioned previously, this solution holds when $f_{10} = 0$, for either $n = 1/2$ or $n = 3/4$. According to equations (15) and (16), the perturbation potential and velocity components are

$$\varphi(x,y) = \frac{1}{3y^{1/2}} \frac{x^3}{(U_o k_o)y^{3/2}} = \frac{1}{3(\gamma+1)} \frac{x^3}{y^2} \tag{33a}$$

$$\varphi_x(x,y) = \frac{1}{(\gamma+1)}\left(\frac{x}{y}\right)^2 , \qquad \varphi_y(x,y) = \frac{-2}{3(\gamma+1)}\left(\frac{x}{y}\right)^3 \tag{33b}$$

Consistent with the approximations made in establishing equation (10), the slope of the streamlines is given by

$$\frac{dy}{dx} = \varphi_y \tag{34}$$

and, after integration, this yields

$$y^4 + \frac{2}{3(\gamma+1)} x^4 = {y_o}^4 \tag{35}$$

where y_o is the starting value of y. The flow therefore expands as it becomes supersonic. The solution given in equations (33) is clearly of the sort called "simple wave" since it lies adjacent to an undisturbed region. The characteristics of the field should then consist of at least one set of straight lines, and this is seen to be the case; for, from equation (14) with $\sigma = 0$, the slopes of the families of characteristics are

$$\frac{dy}{dx} = \frac{\pm 1}{[(\gamma+1)\varphi_x]^{1/2}} \tag{36}$$

and, using equation (33b),

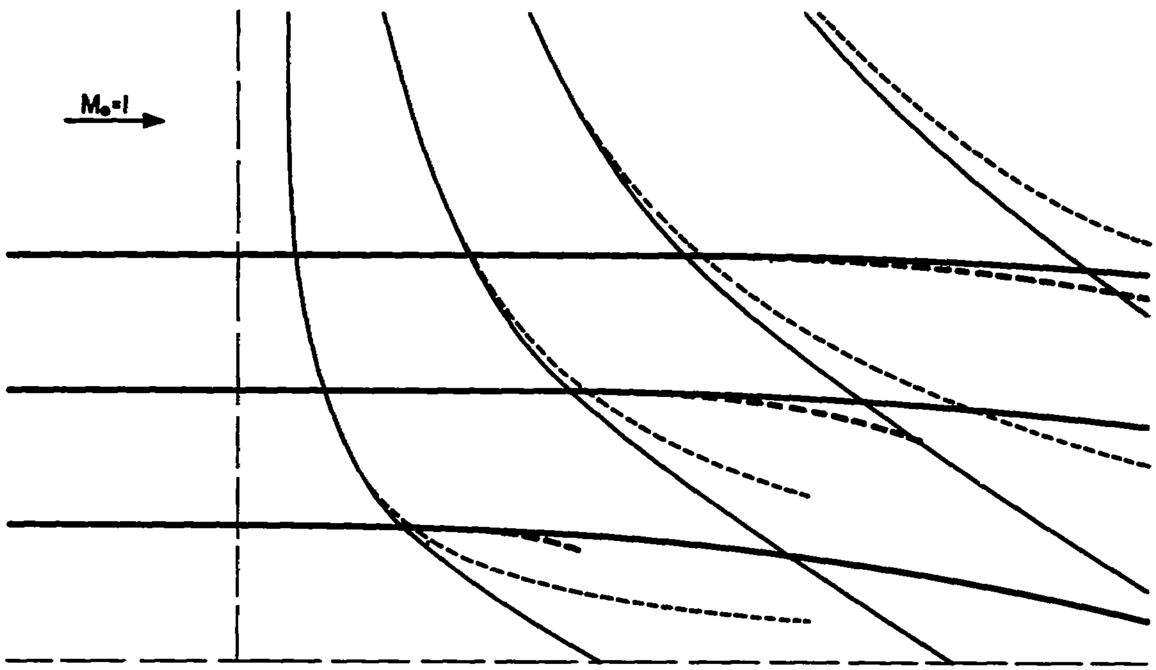
$$\frac{dy}{dx} = \pm \frac{y}{x} \tag{37}$$

The characteristics are then

$$y/x = \text{const} \;, \qquad xy = \text{const} \tag{38}$$

and the first of these families is the required set of straight lines.

The best known simple wave solution is that of the Prandtl-Meyer expansion and, in fact, the present solution is the transonic approximation to this exact solution, applying for small angles of flow deflection. The accompanying sketch (g) shows a comparison of the two flows as to streamlines and second-family characteristics (the first-family characteristics are identical, being radial lines from the origin.)



Sketch (g)

The solid lines are the exact results and the dashed lines represent the present approximation. Roughly speaking, agreement is good up to a fan angle of about 45°, corresponding to a deflection, or expansion, through about 6°. Another comparison that can be made is that of the pressure coefficients in the two solutions. The exact result is (for $M_o = 1$)

$$C_p = \frac{-2}{\gamma}\left(1 - \cos^{\frac{2\gamma}{\gamma-1}}\lambda\omega\right) \tag{39}$$

where γ is the ratio of specific heats, $\lambda^2 = (\gamma - 1)/(\gamma + 1)$ and $\tan \omega = x/y$ so that ω is the polar angle measured from the y axis. For small angles of deflection α, it is easily found that the relation between α and ω is

$$\alpha \approx \frac{2}{3(\gamma+1)}\,\omega^3$$

If the right side of equation (39) is expanded for small values of ω and the lowest-order term retained, one gets

$$C_p \approx \frac{-2}{\gamma+1}\,\omega^2$$

In terms of α, the last relation yields for pressure coefficient

$$C_p \approx \frac{-\alpha^{2/3}(18)^{1/3}}{(\gamma+1)^{1/3}} \tag{40}$$

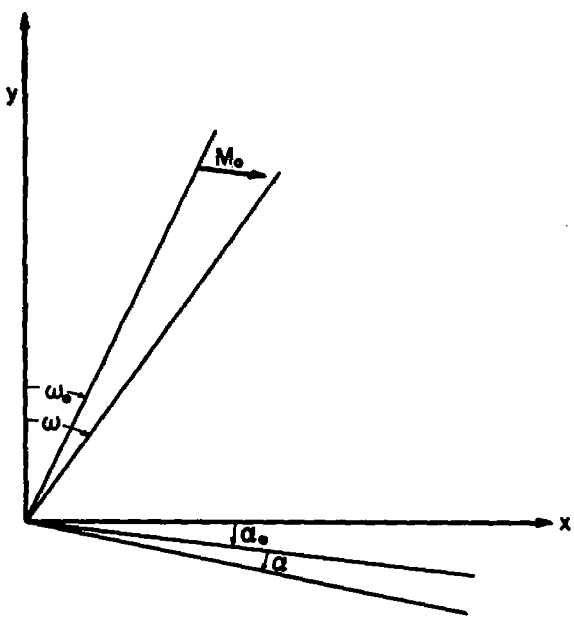
In the present transonic small-perturbation theory, pressure coefficient is

$$C_p = -2\varphi_x = \frac{-2}{(\gamma+1)}\left(\frac{x}{y}\right)^2 = \frac{-2}{(\gamma+1)}\,\omega^2$$

The deflection, α, of the flow is

$$\alpha = -\varphi_y = \frac{2}{3(\gamma+1)}\left(\frac{x}{y}\right)^3 = \frac{2}{3(\gamma+1)}\,\omega^3$$

and these last two relations lead directly to the approximation for Prandtl-Meyer flow given in equation (40). This expression is, moreover, in accordance at $M_o = 1$ with the transonic similarity rule for pressure coefficient as given, for example, in reference 3.



Sketch (h)

The nature of the Prandtl-Meyer relation between pressure and local flow deflection leads naturally to application to two-dimensional expansion over a curved airfoil surface for arbitrary free-stream Mach numbers. The problem resolves itself into the determination, from a knowledge of the expansion fan for $M_o = 1$, of a relation for pressure coefficient, analogous to equation (40), for the case in which a flow at a Mach number M_o is turned through a small angle. To determine such a formula, one starts with the previous solution and follows it until the Mach number in the expansion becomes M_o, at some angle ω_o, as shown in sketch (h). The velocity is constant all along the ray at angle ω_o to the y axis, and the direction of the flow is parallel to the ray at angle α_o to the x axis. Introduce now a new potential function $\psi(x,y)$ defined by

$$\psi(x,y) = -\frac{(M_o{}^2-1)x}{(\gamma+1)M_o{}^2} + \frac{\varphi}{M_o{}^2} \tag{41}$$

This function satisfies the transonic equation (13) for two-dimensional flow at free-stream Mach number M_o, and so applies to the flow in the present problem as it exists after turning through the angle α_o. The boundary conditions are

$$\psi_x = \frac{\omega_o{}^2 - (M_o{}^2-1)}{(\gamma+1)M_o{}^2} \qquad \text{at} \quad \omega_o$$

and since $\omega_o{}^2 = M_o{}^2 - 1$ in the present approximation, one has

$$\psi_x = 0 \qquad \text{at} \quad \omega_o$$

In general, the streamwise component of perturbation velocity is

$$\psi_x = \frac{\omega^2 - \omega_o{}^2}{(\gamma+1)M_o{}^2}$$

The deflection angle is

$$\psi_y = \frac{-2}{3(\gamma+1)}\frac{\omega^3}{M_o{}^2} = -(\alpha_o + \alpha)$$

and the initial boundary condition yields

$$\alpha_o = \psi_y\Big|_{\omega=\omega_o} = \frac{-2\omega_o{}^3}{3(\gamma+1)M_o{}^2} = \frac{-2(M_o{}^2-1)^{3/2}}{3(\gamma+1)M_o{}^2}$$

Thus the incremental pressure coefficient attributable to the deflection from α_o to $\alpha_o + \alpha$ is

$$C_p = -2\psi_x = 2\frac{(M_o{}^2-1)}{(\gamma+1)M_o{}^2}\left(1 - \frac{\omega^2}{\omega_o{}^2}\right)$$

or, if the relation between ω and α is used,

$$C_p = 2\,\frac{(M_o^2 - 1)}{(\gamma + 1)M_o^2}\left\{1 - \left[1 + \frac{3(\gamma + 1)M_o^2}{2(M_o^2 - 1)^{3/2}}\,\alpha\right]^{2/3}\right\} \tag{42}$$

Equation (42) is consistent with the transonic similarity rule for pressure coefficient when $k = (\gamma + 1)M_o^2/U_o$ is the basic transonic parameter. It is of interest to remark that the same problem has been treated by Lighthill in his discussion of simple wave theory of airfoil flow (ref. 6, p. 387). His result, which was derived as a summation of series representations for pressure coefficient in terms of flow deflection angle, was

$$C_p = 2\,\frac{(M_o^2 - 1)}{(\gamma + 1)}\left\{1 - \left[1 - \frac{3(\gamma + 1)}{2(M_o^2 - 1)^{3/2}}\,\alpha\right]^{2/3}\right\} \tag{43}$$

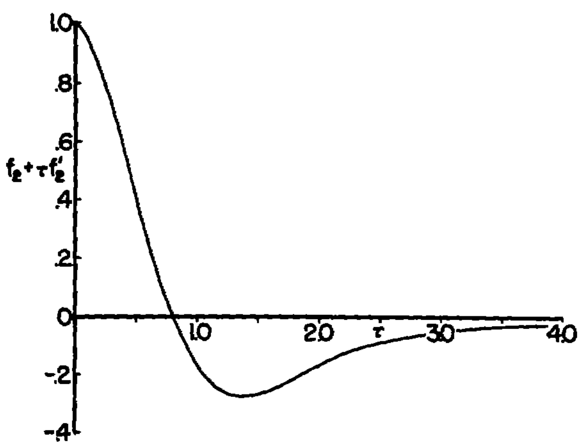
Here, α is of opposite sign than that used in equation (42). It is clear that equation (43) is in a form consistent with the transonic similarity rules based upon the value $(\gamma + 1)/U_o$ of the transonic parameter k.

Subsonic-type solutions.- Consider the two-dimensional solution for the case $n = 1/2$. The differential equation defining the streamlines of the field is

$$\frac{dy}{dx} = \varphi_y = \frac{-1}{2y^{3/2}}\left[f_2(\tau) + \tau f_2'(\tau)\right] \tag{44a}$$

This equation must be solved numerically and for the reason of avoiding continual conversion between x, y, and τ, it is convenient to change the independent variable from x to τ. This gives

$$\frac{dy}{d\tau} = -2y\,\frac{f_2 + \tau f_2'}{4(U_o k_o)^{-1/3}y^2 + \tau(f_2 + \tau f_2')} \tag{44b}$$



Sketch (i)

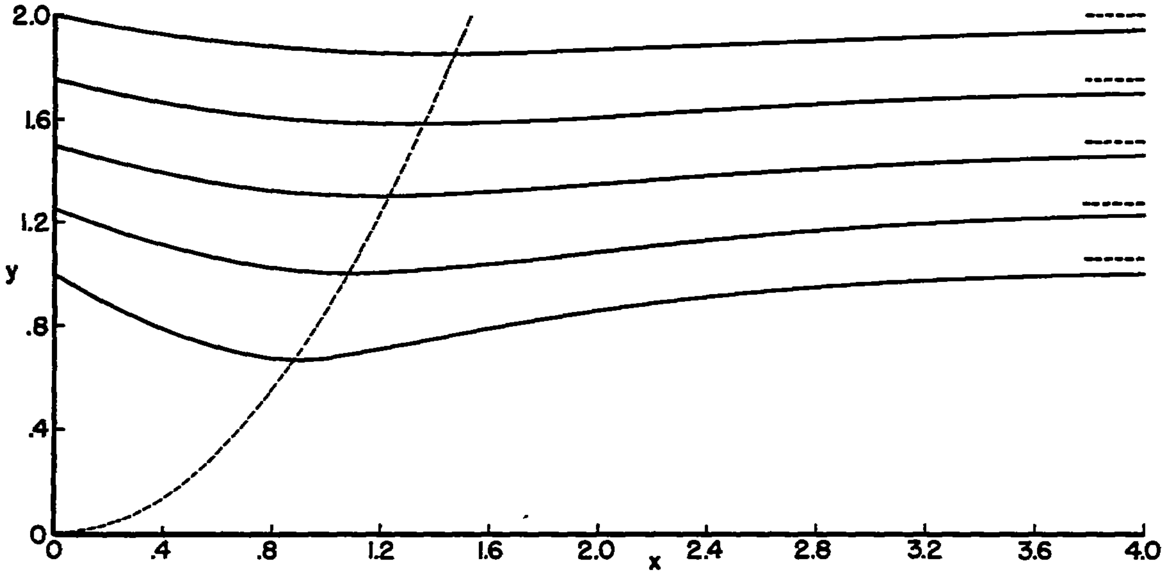
A plot of the quantity $(f_2 + \tau f_2')$ versus τ is shown in sketch (i). The quantity vanishes at one value of τ, say τ_1, and this is easily found to be at the point where $f_2(\tau_1) = \tau_1^3$ (by using eq. (20) for f_2' in terms of f_2) or, using equation (21b), where

$$\tau_1^6 = \frac{9}{32}\,f_{20}^2$$

Thus, the streamlines have a stationary point at the zero of $f_2 + \tau f'$ and, in the x,y plane, a constant value of τ becomes a parabola

$$x^2 = \left[(U_0 k_0)^{1/3} \tau_1\right]^2 y$$

Sketch (j) shows some results of integrating equation (44b), and retransforming the independent variable from τ to x, as well as the parabola on which the minimum points of the streamlines lie. In addition one can derive various other relations for the streamlines as approximations.



Sketch (j)

For example, for very small values of τ (or x), one can use the first term only of the series (24a) for $f_2(\tau)$, and derive the starting form of the curves. This gives

$$y^{5/2} = y_0^{5/2} - \frac{5}{4} f_{20} x$$

where

$$y_0 = y(0)$$

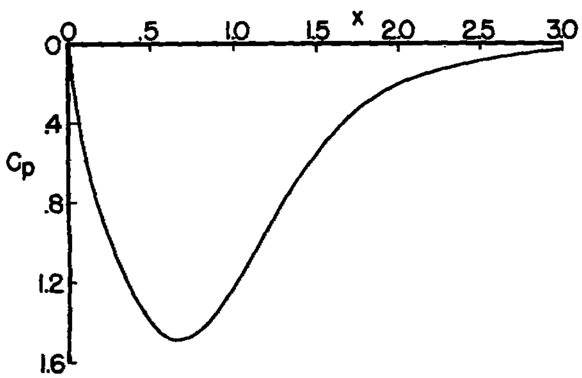
On the other hand, to determine the asymptotic form of the streamlines, the first term of the series (24b) can be used together with equation (44a). Thus

$$y_\infty - y = \frac{3(U_0 k_0)}{8} f_{20}^2 \frac{1}{x^2}$$

where y_∞ is the ultimate height of the streamline. In this particular example, the streamlines approach their asymptotes from below. Finally, for large starting values of the streamlines, $y_o \gg 1$, a control surface method would be applicable for the calculation, since the variation dies out rapidly with increasing y_o, and the streamlines are nearly straight.

It remains to calculate the pressure distribution along a streamline in the flow. The expression for pressure coefficient is given explicitly as

$$C_p = -2\varphi_x = \frac{-2}{(U_o k_o)^{1/3} y} f'(\tau) \tag{45}$$



Sketch (k)

The results of the numerical solution of the differential equation (44b) for the streamlines are useful for this pressure calculation, since therein y is known as a function of τ. Results for the streamlines having $y_o = 1$, based on the function $f_2(\tau)$ for which $f_{2o} = 1$, are shown in sketch (k). It is not difficult to determine that near $x = 0$,

$$\frac{dC_p}{dx} \sim \frac{1}{\sqrt{x}}$$

so that the pressure gradient is infinite at the y axis, on every streamline. This effect is attributable to the finite deflection angle on each streamline at $x = 0$.

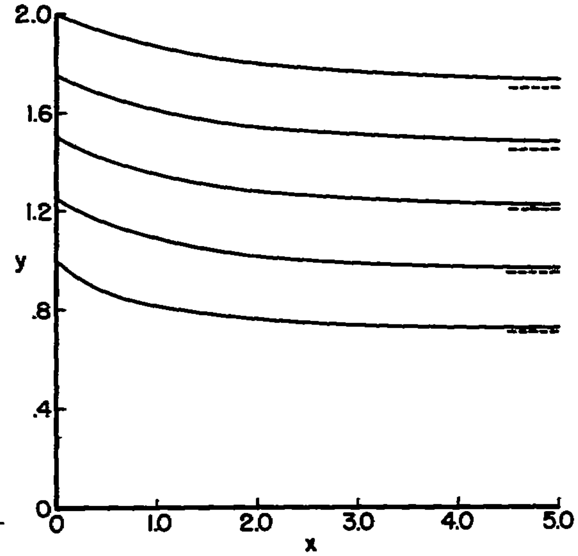
The other two-dimensional solution $(n = 3/4)$ can be discussed in much the same way as for the preceding case $(n = 1/2)$. The differential equation for the streamlines is

$$\frac{dy}{dx} = \frac{-2}{3\tau y^{3/4}} g_2'^2(\tau)$$

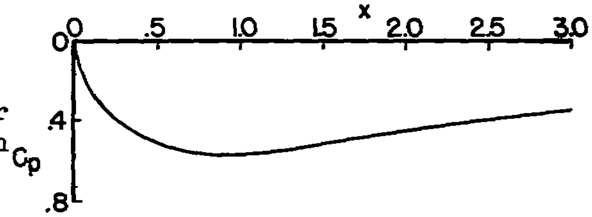
Again, the independent variable can be changed from x to τ, resulting in

$$\frac{dy}{d\tau} = -\frac{4}{3}\frac{y}{\tau} \frac{g_2'^2(\tau)}{2(U_o k_o)^{-1/3} y + g_2'^2(\tau)}$$

Results for numerical solution of this equation, for several initial values of the lateral coordinate y_o, are indicated in sketch (l). The initial value of $g_2(\tau)$ was $g_{2o} = -1$. It is noticed that there are no stationary points on these streamlines, but the corner at $x = 0$ persists. Pressure on the streamline for which $y_o = 1$ is shown in sketch (m), and the singularity in the pressure gradient at $x = 0$ still exists.



Sketch (l)



Sketch (m)

<u>Combination of subsonic and supersonic solutions</u>.- It is of interest to consider the possibility of combining the previous solutions through the mechanism of a shock wave. Thus, consider again the undisturbed sonic flow in the half-plane $x < 0$, and let it expand to become supersonic for $x \geqq 0$. If this supersonic flow is followed for some small distance, it can be shown that a shock wave of determined strength can be specified in such a way that the downstream field corresponds to known results. Assume now that the simple-wave, supersonic solution is to be followed by the subsonic solution for the case $n = 1/2$. The discontinuities in the flow must satisfy the two-dimensional form of the shock-wave relation given in equation (11). Since $M_o = 1$, the condition can be written in the form

$$\left(\varphi_{y_a} - \varphi_{y_b}\right)^2 = (U_o k_o)\ \frac{\varphi_{x_a} + \varphi_{x_b}}{2} \left(\varphi_{x_a} - \varphi_{x_b}\right)^2 \tag{46}$$

In terms of the auxiliary functions $f_1(\tau)$ and $f_2(\tau)$, equation (46) becomes

$$(f_1 - f_2 + \tau f_1' - \tau f_2')^2 - 2(f_1' + f_2')(f_1' - f_2')^2 = 0$$

which factors to give

$$(f_1 - f_2)(f_1 - \tau f_1' - f_2 + \tau f_2') = 0$$

The shock-wave relation is thus satisfied if

$$f_1(\tau) = f_2(\tau) \quad . \tag{47}$$

(The other factor cannot vanish since $f_1 - \tau f_1'$ is a negative quantity and $f_2 - \tau f_2'$ is always positive.) Since equation (47) will hold for some fixed value of τ, say τ_s, it is clear that the shock wave in the physical plane must have a parabolic shape, with the equation

$$x^2 = (U_o k_o)^{2/3} \tau_s{}^2 y \tag{48}$$

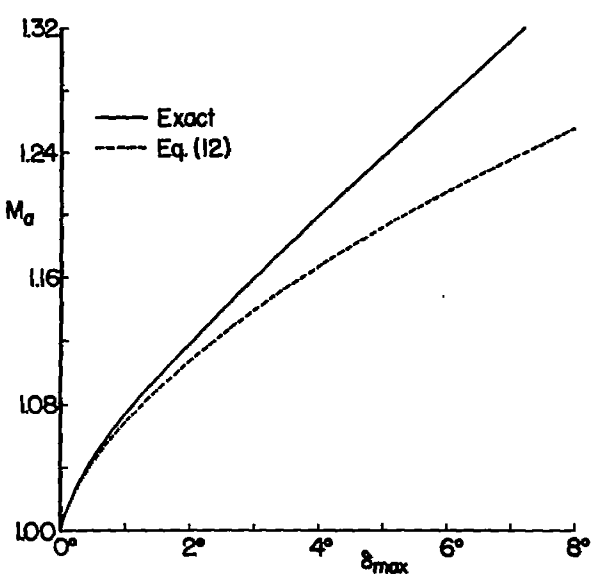
Further, it is not difficult to determine, by setting $f_2(\tau) = \tau^3/3$ in equation (21b), that

$$\tau_s = \left(\frac{4}{3} f_{2o}{}^2\right)^{1/6} \tag{49}$$

In order to see if the condition represented by equation (47) is indeed sufficient for the existence of a shock wave, velocities can be calculated on both sides of the parabola of equation (48) in directions tangent to and normal to the parabola. One then finds that the tangential components are continuous across the parabola and that the normal components fulfill, to the order of transonic small-perturbation theory, the condition

$$(V_n)_a (V_n)_b = 1 - \frac{\gamma - 1}{\gamma + 1} V_t{}^2$$

where V_n and V_t represent components of the total fluid velocity normal and tangential to the shock wave, respectively, and the subscripts a and b refer to conditions immediately upstream and downstream of the shock wave.



Sketch (n)

A final condition to be met, if the joining of the two flow fields by a shock wave is to be meaningful, is that the shock wave be attached to the wall. Equation (12) gives an expression for the shock-attachment angle δ_{max}, and sketch (n) shows a comparison of the exact values of δ_{max} as derived from oblique-shock-wave theory with the approximate values given by equation (12) with $M_o = 1$ as a function of the local Mach number just ahead of the shock wave, M_a. This sketch re-emphasizes the necessity for confining the application of this perturbation analysis, where shock conditions are referred to free-stream quantities, to very small values of the perturbed quantities. Now, to insure that deflection angles obtained in matching the supersonic and subsonic flow fields are admissible, it is necessary to determine the angle at which streamlines of both parts intersect at the shock wave. If λ is the angle of inclination of a streamline, its slope is

$$\tan \lambda = \frac{dy}{dx} = \varphi_y = \frac{f + \tau f'}{-2y^{3/2}}$$

The discontinuity across the shock wave at any point thereof is then, making use of equation (20) and the shock condition (47),

$$\tan(\lambda_b - \lambda_a) = \frac{3}{4y^{3/2}} \frac{\tau_s^3}{1 - \dfrac{\tau_s^6}{18y^3}}$$

$$= \frac{(3^{1/2}/2)(f_{2o}/y^{3/2})}{1 - (2f_{2o}^2/27y^3)} \tag{50}$$

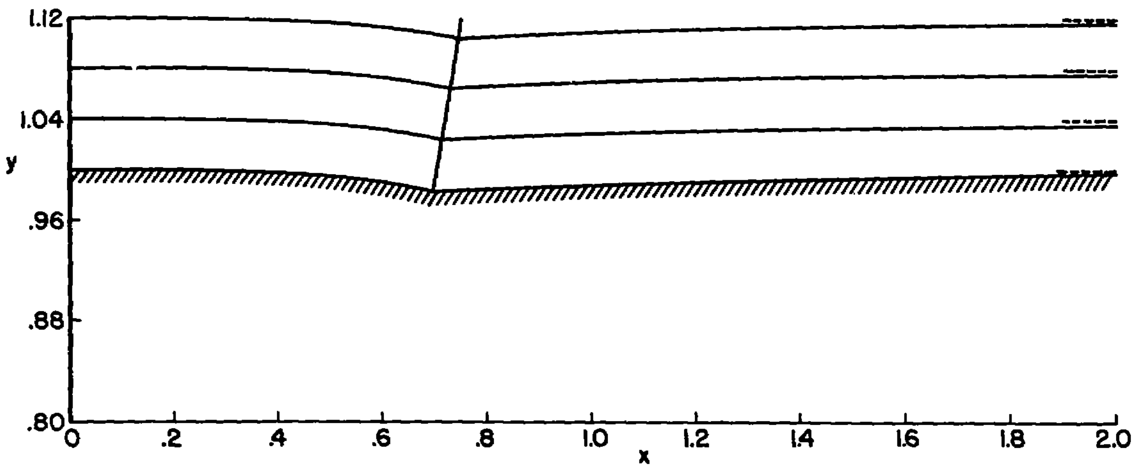
Notice that the deflection decreases with the 3/2 power of distance from the x axis. The angle determined by equation (50) is to be compared with δ_{max}, using as the local Mach number in equation (12), or in sketch (n), that attained at the end of the supersonic portion of the desired combined streamline. The local Mach number M is determined from the transonic relation

$$1 - M^2 \approx 1 - M_o^2 - (U_o k)\varphi_x \tag{51}$$

which approximates the coefficient $(1 - M^2)$ of φ_{xx} in the exact potential equation by the coefficient of φ_{xx} in equation (13) (where $U_o\varphi = \Phi$).

The calculation of an illustrative example is most easily carried out by starting first with one of the subsonic streamlines already calculated and matching to it, at the point of intersection with the shock wave, a supersonic streamline. The latter is readily determined from the differential equation (34) when one uses as the boundary condition the values of x,y obtained from the intersection of the wave and the subsonic streamline.

In order not to extend the supersonic flow to angles of deflection that exceed the range of validity of the theory, it is necessary to choose f_{2o} properly. The results of calculations for which $f_{2o} = 1/8$ are shown in sketch (o). The subsonic streamline was taken as the one



Sketch (o)

for which $y_o = 1$. The shock wave is a portion of the parabola

$$x^2 = (U_o k_o)^{2/3} \tau_s^2 y$$

which becomes, after evaluating τ_s from equation (49),

$$x^2 = 0.493y$$

From equation (50), the calculation to determine change in deflection angle at the shock wave is as follows

$$\tan(\lambda_b - \lambda_a) = 0.111 \;, \qquad \lambda_b - \lambda_a = 6.3^\circ$$

In order to calculate the local Mach number M_a at the end of the expansion region of this flow, we use the relation (51) which, for $M_o = 1$, gives

$$M_a = \left[1 + (\gamma + 1)\varphi_{xa}\right]^{1/2}$$

and, from equation (33b),

$$M_a = \left[1 + \left(\frac{x_s}{y_s}\right)^2\right]^{1/2}$$

With the value of M_a thus determined, sketch (n) can be entered, and it is found that the angle $\lambda_b - \lambda_a$ just determined lies above the shock-attachment limit, so that the shock wave is attached to the wall.

In connection with the wall shape and shock wave of sketch (o), it should be mentioned that the analysis leads to the so-called "strong shock" and that it is not possible, in this two-dimensional case, to derive a shape characterized by a "weak shock." This result can be found by a consideration of the perturbation shock relation (11). The region of the shock polar that lies between the value of φ_{x_b} corresponding to maximum deflection angle and $\varphi_{x_b} = 0$ corresponds to the weak shock region. It is easy to determine that in this region

$$-\frac{\varphi_{x_a}}{3} \leq \varphi_{x_b} \leq 0$$

Now, using equation (16), it can be shown that this condition corresponds, in terms of the variable τ, to

$$\tau^6 \geq \left(\frac{27}{16} f_{20}\right)^2 > \tau_s{}^6$$

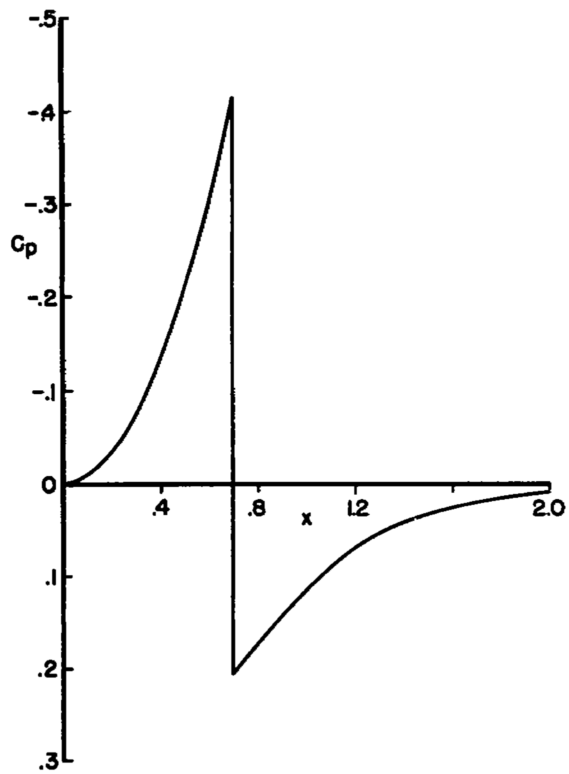
where τ_s is given in equation (49). The parabola in the xy plane for which $\varphi_{x_b} \geq -\left(\varphi_{x_a}/3\right)$ therefore lies outside the shock-wave parabola determined by τ_s. Thus, the weak-shock condition can never be achieved in this case.

On the supersonic portion of the streamline just illustrated, pressure coefficient is, from equation (33b)

$$C_p = -2\varphi_x = -\frac{2}{\gamma+1}\left(\frac{x}{y}\right)^2$$

where y is the height of the streamline, $y = y(x)$. If the point where the shock wave intersects the streamline is denoted (x_s, y_s), pressure coefficient on the upstream side is

$$C_p = -\frac{2}{\gamma+1} \frac{x^2}{\left[y_s{}^4 + \frac{2}{3(\gamma+1)}\left(x_s{}^4 - x^4\right)\right]^{1/2}} \tag{52}$$



Sketch (p)

In the subsonic portion of the field, calculation of pressure is not so simple, but is given by

$$C_p = -\frac{2}{(\gamma+1)^{1/3}y} f_2'(\tau)$$

where $y = y(x)$ is the calculated height of the subsonic streamline. Results of the pressure calculations on the wall shown in sketch (o) are given in sketch (p).

It is not possible to combine so directly the subsonic and supersonic flows when the other two-dimensional solution, $g(\tau)$, is to be used. The reason is that since the condition for a shock wave is again $g_1 = g_2$, the simple supersonic-type solution $g_1 = \tau^3/3$ is not applicable since the subsonic solution g_2 is always negative and no intersection occurs. Further application of these results would involve detailed examination of the more general supersonic solution given by equation (26a).

Axially Symmetric Flow

Supersonic solution.- A single solution exists in the axially symmetric case, corresponding to the value 1/2 for the index n. There are, of course, still the solutions of supersonic type and of subsonic type, as given in equations (30) and (31). A simple supersonic solution exists again, namely, from equation (31)

$$F_1(\tau) = \frac{2}{9}\tau^3$$

The corresponding potential and velocity functions are

$$\varphi = \frac{1}{y^{1/2}} F_1(\tau) \quad , \qquad \varphi = \frac{2}{9(U_o k_o)} \frac{x^3}{y^2}$$

$$\varphi_x = \frac{2}{3(U_o k_o)} \left(\frac{x}{y}\right)^2 \quad , \qquad \varphi_y = -\frac{4}{9(U_o k_o)} \left(\frac{x}{y}\right)^3 \qquad (53)$$

so that certain aspects of this flow are similar to those in the two-dimensional supersonic case previously considered. A definite difference does exist, however, in the fact that the two-dimensional streamlines could be carried to the limit as y approaches zero and resulting in this case in a Prandtl-Meyer corner flow. This limiting process fails three-dimensionally since the flow turns inward on itself. Finite values of streamline radius are therefore required and the flow must be modified downstream by a shock wave and a readjustment of the streamlines.

Subsonic solution.- The subsonic solution for the axially symmetric transonic problem depends on the function $F_2(\tau)$ defined in equation (30b) and shown in sketch (f). Streamlines corresponding to this case can again be calculated by solving the differential equation

$$\frac{dy}{dx} = -\frac{1}{2y^{3/2}}\left[F_2(\tau) + \tau F_2'(\tau)\right]$$

By means of the differential equation (28), this can be written

$$\frac{dy}{dx} = -\frac{(U_o k_o)^{1/3}}{xy} F_2'^2(\tau) \tag{54a}$$

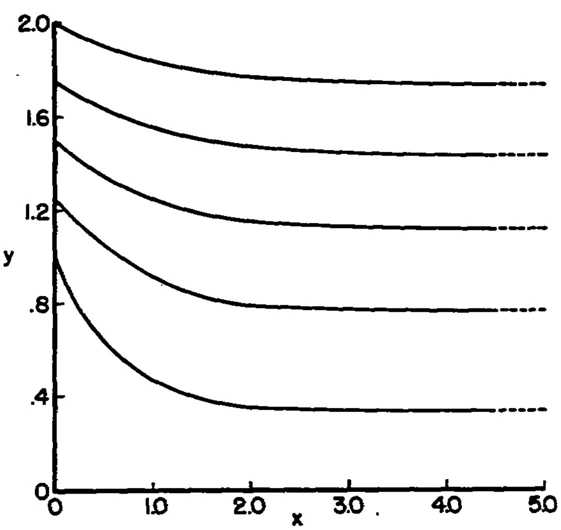
Again, it is convenient to change independent variables from x to τ, and the resulting equation is

$$\frac{dy}{d\tau} = -\frac{2y}{\tau}\,\frac{F_2'^2(\tau)}{2(U_o k_o)^{-1/3}y^2 + F_2'^2(\tau)} \tag{54b}$$

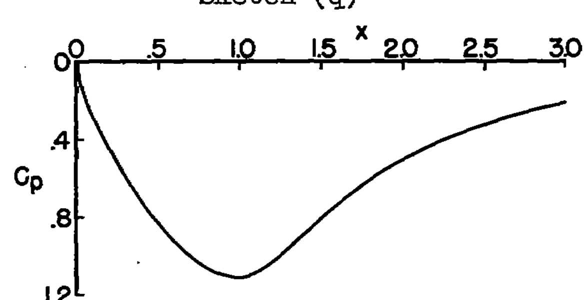
In this case, there are no stationary points on the streamlines. A few results of calculations are shown in sketch (q) (where F_{2o} has been taken equal to 1). These streamlines approach their asymptotic height from above, as in the two-dimensional case with $n = 3/4$, rather than from below, as in the two-dimensional case with $n = 1/2$. Pressure calculations proceed as previously, the formula being

$$C_p = -\frac{2}{(\gamma+1)^{1/3}y} F_2'(\tau)$$

In sketch (r), the pressure on the streamline for which $y_o = 1$ is shown.



Sketch (q)



Sketch (r)

Combined supersonic and subsonic solutions.- For the axially symmetric problem, it is again possible to use the simplest supersonic function $F_1(\tau) = (2/9)\tau^3$ in conjunction with the subsonic solution $F_2(\tau)$, as given by equation (30b), to produce a flow in which a shock wave is imbedded. The condition for existence of a shock wave in the axially symmetric case is, from equation (11)

$$\left(\varphi_{y_a} - \varphi_{y_b}\right)^2 = (U_o k_o)\left(\frac{\varphi_{x_a} + \varphi_{x_b}}{2}\right)\left(\varphi_{x_a} - \varphi_{x_b}\right)^2$$

where y now denotes the radial coordinate. In terms of the auxiliary function $F(\tau)$, this last condition can be written

$$(F_1 - F_2)(F_1'^2 - F_2'^2) = 0 \tag{55}$$

Again it can be shown that the first of these factors leads to a proper relation of tangential and normal velocities across the shock wave, while the second does not. The condition is, therefore, that the functions $F_1(\tau)$ and $F_2(\tau)$ be equal at the shock wave. The value of τ at which this occurs is

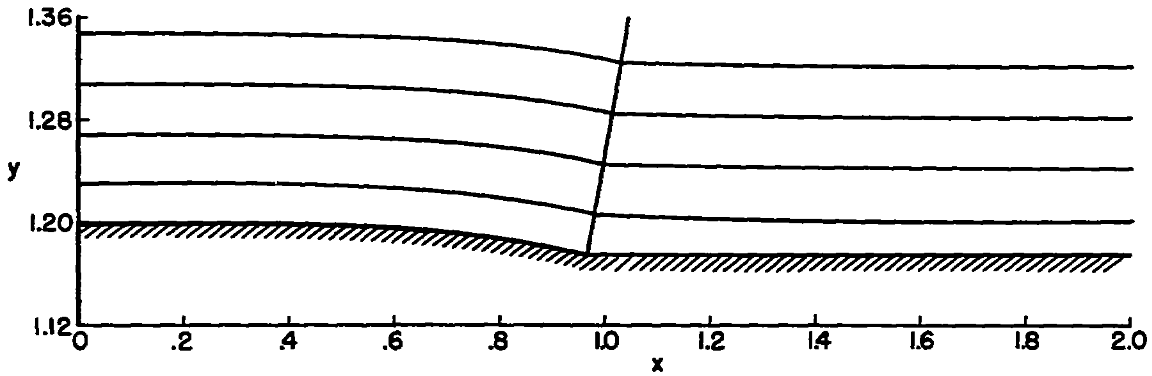
$$\tau_s^4 = \frac{36}{5}\left(\frac{9F_{2o}^4}{100}\right)^{1/3} \tag{56a}$$

and the shock wave is then given in the physical plane by

$$x^2 = \left[(U_o k_o)^{1/3}\tau_s\right]^2 y \tag{56b}$$

To illustrate the sort of body that one gets in this case, consider a streamline (subsonic) for which $y_o = 1.2$, and computed on the basis $F_{2o} = 1/8$. The procedure is the same as it was previously, namely to determine the intersection, say (x_s,y_s), of this streamline with the parabola (eq. (56b)) representing the shock wave, and to use this point as a boundary condition for determining the proper solution to the supersonic streamline differential equation

$$\frac{dy}{dx} = -\frac{4}{9(U_o k_o)}\left(\frac{x}{y}\right)^3 \tag{57}$$



Sketch (s)

The result of such a calculation, with the parameters y_o and F_{20} taken as mentioned, gives the figure shown in sketch (s). The angle between streamlines on the two sides of the shock wave is, in this case

$$\lambda_b - \lambda_a = 5.7^o$$

In this axially symmetric case one gets by using equation (51) with $M_o = 1$,

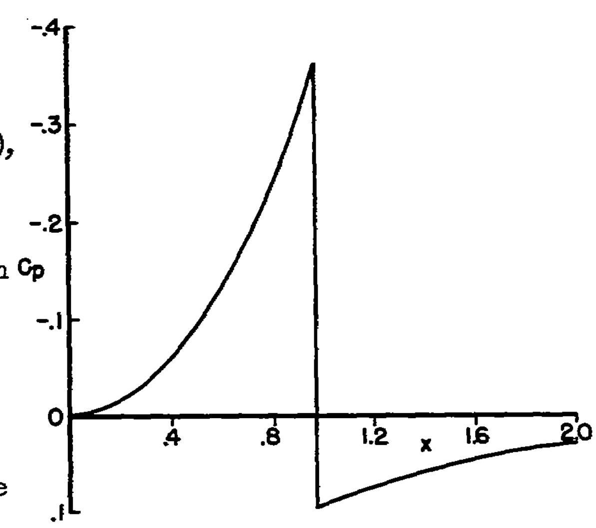
$$M_a^2 = 1 + \frac{2}{3}\left(\frac{x_s}{y_s}\right)^2$$

and calculation of M_a shows that the shock wave is attached in this case. In contrast to the two-dimensional case treated above, it is found that in the present axially symmetric case it is necessary that the shock wave be of the weak family, because the critical parabola is determined by

$$\tau^4 = \frac{243}{128} F_{20}\left(\frac{3}{2} F_{20}\right)^{1/3} < \tau_s^4$$

where τ_s is given in equation (56), and therefore lies inside the shock wave. Since in both the two-dimensional case and the present one, the line of demarcation between strong and weak shocks is distinct from the shock wave itself, one cannot determine configurations having shock waves that are right at the point of detachment.

The pressure coefficient on the streamlines of sketch (s) is shown in sketch (t).



Sketch (t)

Mass-flow property of subsonic solution with axial symmetry.- All of the solutions given here were derived by means of formal manipulation of the transonic differential equation but in the case of the supersonic solutions $f_1(\tau) = \tau^3/3$, $g_1(\tau) = \tau^3/3$, and $F_1(\tau) = (2/9)\tau^3$ it was easy to identify these results with simple expansions from uniform stream conditions. It is evident that the latter solutions could have been derived from any one of several methods of attack since their existence and nature are intuitively so obvious. The subsonic solutions, on the other hand, are somewhat more complex and appear to arise rather artificially as a result of abrupt deflections imposed on the streamlines across the entire plane of flow. In the case of axial symmetry, however, the subsonic solution has properties reminiscent of the fundamental solution identified with the source-sink potential of linearized supersonic theory. From equations (15) and (16) the perturbation potential and velocity components can be rewritten as

$$\varphi = \frac{1}{y^{1/2}} F_2(\tau) \tag{58}$$

$$u = \varphi_x = \frac{F_2'(\tau)}{y(\gamma+1)^{1/3}} ; \qquad v = \varphi_y = -\frac{1}{2y^{3/2}} \frac{d}{d\tau} [\tau F_2(\tau)] \tag{59}$$

From equations (32), limiting conditions yield:

For $\tau \ll 1$

$$u \approx -\frac{1}{(\gamma+1)^{1/3}} \left(\frac{F_{20}}{2}\right)^{1/2} \frac{\tau^{1/2}}{y} + \ldots = -\frac{1}{(\gamma+1)^{1/3}} \left(\frac{F_{20}}{2}\right)^{1/2} \frac{x^{1/2}}{y^{3/4}} + \ldots$$

$$v \approx -\frac{1}{2y^{3/2}} + \ldots$$

For $\tau \gg 1$

$$u \sim -\frac{3}{8} (9F_{20}{}^4)^{1/3} \frac{1}{\tau^2 y} + \ldots = -\frac{3}{8} (9F_{20}{}^4)^{1/3} \frac{1}{x^2} + \ldots$$

$$v \sim -\frac{27}{64} (3F_{20}{}^8)^{1/3} \frac{1}{\tau^5 y^{3/2}} + \ldots = -\frac{27}{64} (3F_{20}{}^8)^{1/3} \frac{y}{x^5} + \ldots$$

It is apparent that the origin is an essential singularity of the flow field, the rate of growth of the velocity components being determined in part by the particular parabola $\tau = \text{const}$ along which one approaches the origin. Away from the origin and on the lateral axis u is zero and

v is finite but attenuating (as $1/y^{3/2}$) with distance. On the longitudinal axis v is zero and u is finite but attenuating (as $1/x^2$) with distance. The flow thus appears to flow into the origin and to distort the field in the portion of the plane downstream of the Mach line at $x = 0$.

It remains to calculate the rate of mass flow that is associated with the solution; if this rate is constant, the analogy with the source is more closely established. To this end, consider a cylinder of fixed radius Y and extending from the plane $x = 0$ to $x = \infty$. The rate of flow is then given by the integral

$$R = \lim_{l \to \infty} \rho_0 \int_0^l 2\pi Y[v]_{y=Y} dx$$

$$= - \lim_{l \to \infty} \int_0^l 2\pi\rho_0 \frac{d[\tau F_2(\tau)]}{d\tau} d\left(\frac{x}{Y^{1/2}}\right)$$

Since $d(x/Y^{1/2}) = (\gamma+1)^{1/3} d\tau$, the integration is direct and yields

$$R = -2\pi\rho_0(\gamma+1)^{1/3} \lim_{l \to \infty} lF_2(l) = -\frac{3}{4}\pi\rho_0(\gamma+1)^{1/3}(9F_{20}{}^4)^{1/3}$$

The rate of mass flow is therefore a negative constant independent of Y and the solution behaves macroscopically like a sink. As seen in sketch (q), the streamlines of the field are drawn toward the longitudinal axis, consistent with this concept.

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